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DETERMINATION OF PLATE TEMPERATURE IN CASE OF COMBINED CONDUCTION, CONVECTION AND RADIATION HEAT EXCHANGE

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Abstract—The singular, integro-differential equation for the temperature of a flat plate with internal energy generation and a fluid flowing over one of its faces, is solved numerically using the method of iteration. The present results compare well with those of Sparrow and Lin, except for the leading portions of the plate. It is also seen that the relations given by Cess for similar problems may not give converged solutions for all cases. The importance of conduction in a plate of high thermal conductivity and of radiation in cases of laminar flow has also been demonstrated.

NOMENCLATURE

a,	constant, $0 < a \leqslant 1$;
$a_1, a_2 \ldots$,	constants;
b,	constant, $0 < b < 1$;
$b_0, b_1, b_2 \ldots$,	constants;
C_1,	constant for a laminar flow, $0{\cdot}623\, Pr^{-\frac{1}{3}}\, Re_L^{-\frac{1}{2}}$;
C_t,	constant for a turbulent flow, $3{\cdot}323\, Pr^{-0{\cdot}6}\, Re_L^{-0{\cdot}8}$;
F_z,	$Y\ddot{\theta}_z - \dfrac{\theta_z^4}{N}$;
h_x,	convective heat-transfer coefficient;
H_z,	$Y\ddot{\theta}_z - \dfrac{\theta_z^4}{N} + \dfrac{1+\Phi}{N}$;
i,	$i \leqslant j$, $1, 2, 3, \ldots, n+1$;
j,	$1, 2, 3, \ldots, n+1$;
k_f,	thermal conductivity of the fluid;
k_p,	thermal conductivity of the plate;
L,	length of plate;
n,	a positive integer;
N,	nondimensional thermal conductivity parameter, $\dfrac{k_p}{\epsilon\sigma T_\infty^3 L}$;
p,	dummy variable for z;
Pr,	Prandtl number of the fluid;
q,	volumetric internal energy generation in the plate;
$\dot{q}_x''$,	convective heat-transfer rate per unit area at the surface;
Re_L,	Reynolds number based on length L, $(U_\infty L/\nu)$;
Re_x,	Reynolds number based on length x, $(U_\infty x/\nu)$;
S_{ij},	series for a laminar flow, $\dfrac{1}{j-1} + \dfrac{8}{21}\dfrac{i^{\frac{7}{4}} - (i-1)^{\frac{7}{4}}}{(j-1)^{\frac{7}{4}}} + \dfrac{2}{9}\dfrac{i^{\frac{5}{2}} - (i-1)^{\frac{5}{2}}}{(j-1)^{\frac{5}{2}}} + \ldots$;
S_{ij}^t,	series for a turbulent flow, $\dfrac{1}{j-1} + \dfrac{80}{171}\dfrac{i^{\frac{19}{10}} - (i-1)^{\frac{19}{10}}}{(j-1)^{\frac{19}{10}}} + \dfrac{160}{567}\dfrac{i^{\frac{14}{5}} - (i-1)^{\frac{14}{5}}}{(j-1)^{\frac{14}{5}}} + \ldots$;

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t,	thickness of the plate;
T_x,	temperature of the plate surface;
T_∞,	freestream temperature;
U_∞,	freestream velocity of the fluid;
x,	length co-ordinate for the plate, $0 \leqslant x \leqslant L$;
X,	nondimensional plate length, $x/L, 0 \leqslant X \leqslant 1$;
Y,	nondimensional plate thickness, t/L;
Z,	nondimensional dummy variable for length, $\xi/L, 0 \leqslant Z \leqslant X$.

Greek letters

β,	$\beta(l, m)$, represents the Beta function of l and m;
Γ,	$\Gamma(l)$, represents the Gamma function of (l);
ΔX,	$1/n$;
ϵ,	hemispherical total emissivity of the plate;
$\dot{\theta}, -, \ddot{\theta}$,	1st, 2nd and 3rd derivatives of nondimensional temperature;
θ_x,	nondimensional temperature, T_x/T_∞;
θ_z,	T_ξ/T_∞;
ν,	kinematic viscosity of the fluid;
ξ,	dummy variable for the plate length, $0 \leqslant \xi \leqslant x$;
σ,	Stefan–Boltzmann constant, 5680×10^{12} W/cm^2 °K^4;
Φ,	nondimensional energy generation parameter, $qt/\epsilon\sigma T_\infty^4$;
Ψ,	nondimensional length parameter, (k_p/k_f) $[1/N(Re_L)]$ $a(X)^b$.

1. INTRODUCTION

With the increasing use of complex thermal systems, analysis of coupled problems becomes very important. In most such systems, heat exchange by only two of the three possible modes is considered. Some problems of combined conduction and radiation have been examined. Viskanta and Grosh [1] analyzed heat transfer by simultaneous conduction and radiation in a gas between two parallel plates. The nonlinear integro-differential equation was solved numerically by an iterative method after reducing it to a nonlinear Fredholm integral equation of the second kind. Howell [2] solved a combined conduction and radiation problem by a finite-difference technique, considering the radiant exchange terms involved in the equation to be independent of the conduction process. Doornink and Hering [3] gave numerical solutions to the transient simultaneous conductive and radiative transfer in a plane gray medium bounded by black walls. The singular nonlinear integro-partial differential equation was solved by representing its nonlinear function by a finite expansion in terms of elementary functions.

Other coupled problems of heat transfer have also received some attention. Oliver and McFadden [4] solved the problem of simultaneous convection and radiation in a laminar boundary layer on an isothermal flat plate by reducing the governing equations to the familiar equation of Blasius. Sparrow and Lin [5] carried out an analysis to determine the distribution of surface temperature on a flat plate undergoing heat exchange with the environment by both convection and radiation and having an internal heat source or sink. The nonlinear integral equation was solved numerically by changing the integral into a series summation and using a predictor–corrector numerical technique. Cess [6, 7] presented an analysis to determine the influence of radiation heat transfer upon the forced convection Nusselt number. Though the solutions presented by Cess do not converge for all the values of plate length, the results were used to find under what conditions radiation may be neglected.

The present study is aimed at determining the temperature profile of a thermal system involving internal energy generation and conduction. Externally, heat is rejected to a flowing transparent gas by convection and to constant temperature black surroundings by radiation. Similar problems are encountered in the design of aircraft and missiles, hot wire anemometry, cooling of electronic instruments and other areas.

2. ANALYSIS OF THE PLATE TEMPERATURE DISTRIBUTION

Derivation of the governing equations

Consider a very thin flat plate of finite length and infinite width with uniform internal generation of thermal energy. Let there be a flow of transparent gas over one face of the plate and let the other face be insulated (see Fig. 1). It is assumed, here, that the thermal conductivity of

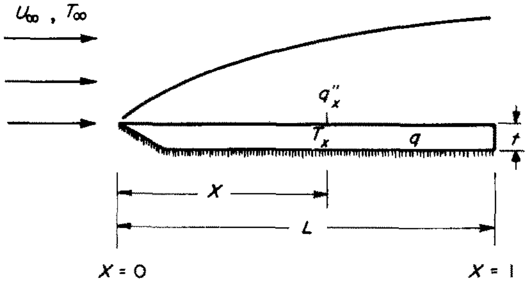

FIG. 1. Thermal model, a flat plate.

the plate and other properties of the plate and air remain constant along the length of the plate, i.e. the properties are independent of any temperature variation. The plate is taken to be a gray surface and the surroundings are considered to be black. The governing energy equation in the plate is obtained by considering an infinitesimal element dx of the plate at a distance x from its leading edge. This analysis yields

$$\dot{q}''_x = h_x(T_x - T_\infty) = k_p t \frac{d^2 T_x}{dx^2} - \epsilon\sigma(T_x^4 - T_\infty^4) + qt, \quad (1)$$

where $\dot{q}''_x$ is the convective heat-transfer rate per unit area of the surface, h_v is the convective heat-transfer coefficient, T_x is the plate temperature, T_∞ is the freestream temperature, k_p is the thermal conductivity of the plate, t is the plate thickness and q is the volumetric internal energy generation in the plate.

For a constant freestream flow along a semi-infinite plate, a solution can be obtained for the wall temperature for the case of arbitrary specified surface (convective) heat flux from Kays' text [8]. Considering a laminar flow

$$T_x - T_\infty = \frac{0{\cdot}623}{k_f} Pr^{-\frac{1}{3}} Re_x^{-\frac{1}{2}} \int_0^x \left[1 - \left(\frac{\xi}{x}\right)^{\frac{3}{4}}\right]^{-\frac{2}{3}} \dot{q}''_\xi \, d\xi, \quad (2)$$

where k_f, Pr and Re are the thermal conductivity, Prandtl number and Reynolds number respectively and ξ is the dummy variable.

Combining the equations (1) and (2) and further simplifying gives

$$T_x - T_\infty = \frac{0{\cdot}623}{k_f} Pr^{-\frac{1}{3}} Re_x^{-\frac{1}{2}} \int_0^x \left[1 - \left(\frac{\xi}{x}\right)^{\frac{3}{4}}\right]^{-\frac{2}{3}} \cdot \left[k_p t \frac{d^2 T_\xi}{d\xi^2} - \epsilon\sigma(T_\xi^4 - T_\infty^4) + qt\right] d\xi.$$

Define the various dimensionless numbers as follows:

$$\frac{x}{L} = X \qquad \frac{\xi}{L} = Z$$

$$\frac{T_x}{T_\infty} = \theta_x \qquad \frac{T_\xi}{T_\infty} = \theta_z$$

$$\frac{t}{L} = Y \qquad 0{\cdot}623\, Pr^{-\frac{1}{3}} Re_L^{-\frac{1}{2}} = C_1$$

$$\frac{k_p}{\epsilon\sigma T_\infty^3 L} = N \qquad \frac{qt}{\epsilon\sigma T_\infty^4} = \Phi.$$

Thus the nondimensional governing equation for the plate temperature in a laminar flow is

$$\theta_x = 1 + C_1 \frac{k_p}{k_f} X^{-\frac{1}{2}} \int_0^X \left[1 - \left(\frac{Z}{X}\right)^{\frac{3}{4}}\right]^{-\frac{2}{3}} \times \left[Y\ddot{\theta}_z - \frac{\theta_z^4}{N} + \frac{1 + \Phi}{N}\right] dZ. \quad (3)$$

This is a singular, integro-differential equation with a nonlinearity in temperature. The kernel

$$H_z\left[1-\left(\frac{Z}{X}\right)^{\frac{3}{4}}\right]^{-\frac{1}{3}},$$

where

$$H_z=\left(Y\ddot{\theta}_z-\frac{\theta_z^4}{N}+\frac{1+\Phi}{N}\right),$$

makes the equation singular as it has a weak singularity at $Z = X$. But the integral exists and converges to a finite value. By removing the singularity, the kernel can be transformed into a continuous function.

In a similar manner the nondimensional governing equation for turbulent flow can be obtained, using a relation given in reference [8].

$$\theta_x=1+C_t\frac{k_p}{k_t}X^{-0.8}\int_0^X\left[1-\left(\frac{Z}{X}\right)^{\frac{9}{10}}\right]^{-\frac{8}{9}}\left[Y\ddot{\theta}_z-\frac{\theta_z^4}{N}+\frac{1+\Phi}{N}\right]\mathrm{d}Z. \qquad (4)$$

where

$$C_t = 3{\cdot}323\,Pr^{-0{\cdot}6}\,Re_L^{-0{\cdot}8}.$$

Closed form analytical solutions to equations of the type (3) and (4) are not known. Therefore they are solved by employing two numerical methods.

Simplification obtained on integrating by parts

The equation (3), for laminar flow, can be rewritten by integrating the right hand side by parts. Thus,

$$\theta_x=1+C_1\frac{k_p}{k_f}X^{-\frac{1}{2}}\times\left\{(H_z)_0^X\int_0^X\left[1-\left(\frac{Z}{X}\right)^{\frac{3}{4}}\right]^{-\frac{1}{3}}\mathrm{d}Z-\int_0^X\frac{\mathrm{d}H_z}{\mathrm{d}Z}\int\left[1-\left(\frac{Z}{X}\right)^{\frac{3}{4}}\right]^{-\frac{1}{3}}\mathrm{d}Z\,\mathrm{d}Z\right\}.$$

It can be easily seen that the term

$$\int[1-(Z/X)^{\frac{3}{4}}]^{-\frac{1}{3}}\,\mathrm{d}Z$$

is equivalent to

$$\int_0^z[1-(p/X)^{\frac{3}{4}}]^{-\frac{1}{3}}\,\mathrm{d}p,$$

where p is a dummy variable of integration. Therefore, it is obtained,

$$\theta_x=1+C_1\frac{k_p}{k_f}X^{-\frac{1}{2}}\left\{\left[\left(Y\ddot{\theta}_x-\frac{\theta_x^4}{N}+\frac{1+\Phi}{N}\right)-\left(Y\ddot{\theta}-\frac{\theta_0^4}{N}+\frac{1+\Phi}{N}\right)\right]\int_0^X\left[1-\left(\frac{Z}{X}\right)^{\frac{3}{4}}\right]^{-\frac{1}{3}}\times\mathrm{d}Z-\int_0^X\left(Y\dddot{\theta}_z-\frac{4\theta_z^3\dot{\theta}_z}{N}\right)\times\int_0^Z\left[1-\left(\frac{p}{X}\right)^{\frac{3}{4}}\right]^{-\frac{1}{3}}\mathrm{d}p\,\mathrm{d}Z\right\}. \qquad (5)$$

Applying equation (1), for the limiting case of $x \to 0$,

$$[T_x-T_\infty]_{x\to 0}=\frac{[k_p t\,\mathrm{d}^2T/\mathrm{d}x^2-\epsilon\sigma(T_x^4-T_\infty^4)+qt]x\to 0}{[h_x]_{x\to 0}}.$$

For a flat plate, the local convective heat-transfer coefficient varies as $1/(x)^{\frac{1}{2}}$, so as

$$x\to 0,\qquad h_x\to\infty,$$

or

$$\theta_0\to 1.$$

Again considering the energy equation (1) for $x \to 0$,

$$h_x(T_x-T_\infty)|_{x\to 0}=k_p t\left.\frac{\mathrm{d}^2T_x}{\mathrm{d}x^2}\right|_{x\to 0}-\epsilon\sigma(T_x^4-T_\infty^4)|_{x\to 0}+qt|_{x\to 0},$$

or

$$Y\ddot{\theta}_0 = -\frac{\Phi}{N}.$$

Therefore, equation (5) reduces to

$$\theta_x = 1 + C_1\frac{k_p}{k_f}X^{-\frac{1}{2}}\left\{\left(Y\ddot{\theta}_x - \frac{\theta_x^4}{N} + \frac{1+\Phi}{N}\right)\right.$$
$$\times\int_0^X\left[1-\left(\frac{Z}{X}\right)^{\frac{3}{4}}\right]^{-\frac{2}{3}}\mathrm{d}Z - \int_0^X\left(Y\dddot{\theta}_z - \frac{4\theta_Z^3\dot{\theta}_z}{N}\right)$$
$$\left.\int_0^Z\left[1-\left(\frac{p}{X}\right)^{\frac{3}{4}}\right]^{-\frac{2}{3}}\mathrm{d}p\,\mathrm{d}Z\right\}. \quad (6)$$

The finite value of a singular integral of the type

$$\int_0^X[1-(Z/X)^a]^{-b}\,\mathrm{d}Z$$

(where $0 < a \leqslant 1$ and $0 < b < 1$), can be obtained by making use of a Beta function. Thus,

$$\int_0^X\left[1-\left(\frac{Z}{X}\right)^a\right]^{-b}\mathrm{d}Z = \frac{X}{a}\beta\left(1-b, \frac{1}{a}\right).$$

[$\beta(l, m)$ represents the Beta function of l and m]

Thus when the integral

$$\int_0^Z[1-(p/X)^{\frac{3}{4}}]^{-\frac{2}{3}}\,\mathrm{d}p$$

of equation (6) becomes singular, it can also be simplified by using a beta function for $Z = X$.

Thus the simplified governing equation for the plate temperature in a laminar flow becomes,

$$\theta_x = 1 + C_1\frac{k_p}{k_f}X^{-0\cdot5}\left\{3{\cdot}530\,X\left(Y\ddot{\theta} - \frac{\theta_z^4}{N}\right.\right.$$
$$\left. + \frac{1+\Phi}{N}\right) - \int_0^X\left(Y\ddot{\theta}_z - \frac{4\theta_z^3\dot{\theta}_z}{N}\right)$$
$$\left.\times\int_0^Z\left[1-\left(\frac{p}{X}\right)^{\frac{3}{4}}\right]^{-\frac{2}{3}}\mathrm{d}p\,\mathrm{d}Z\right\}. \quad (7)$$

If the flow is turbulent, the governing equation (4) can be simplified by following a procedure similar to the laminar flow case, to obtain,

$$\theta_x = 1 + C_t\frac{k_p}{k_f}X^{-0\cdot8}\left\{9{\cdot}827\,X\left(Y\ddot{\theta}_z - \frac{\theta_z^4}{N}\right.\right.$$
$$\left. + \frac{1+\Phi}{N}\right) - \int_0^X\left(Y\ddot{\theta}_z - \frac{4\theta_z^3\dot{\theta}_z}{N}\right)$$
$$\left.\times\int_0^Z\left[1-\left(\frac{p}{X}\right)^{\frac{9}{10}}\right]^{-\frac{8}{9}}\mathrm{d}p\,\mathrm{d}Z\right\}. \quad (8)$$

Simplification is obtained by expanding in a series.

The integral in the governing equation (3) can be simplified by using another method as illustrated by Sparrow and Lin [5]. The treatment presented by Sparrow and Lin is only for the case where the delay factor in the kernel is of the type $[1-(Z/X)]^{-b}$ and not of the type $[1-(Z/X)^a]^{-b}$ as used in the present analysis. The simplification in the form of delay factor used by Sparrow and Lin was obtained, as proposed by Hanna and Mayers [9], by using a superposition of step-changes in surface heat flux instead of a superposition of step-changes in surface temperature.

Divide the region between $X = 0$ and $X = 1$ into n equal parts, such that $\Delta X = 1/n$. Any value of X in the region $0 \leqslant X \leqslant 1$ can be denoted by $X = (j-1)\Delta X$ and $Z = (i-1)\Delta X$. Also, in equation (3) for $|Z/X| < 1$, the kernel can be expanded in a binomial series. Thus

$$\int_0^X F_z\left[1-\left(\frac{Z}{X}\right)^{\frac{3}{4}}\right]^{-\frac{2}{3}}\mathrm{d}Z = \sum_{i=1}^{j}$$
$$\int_{(i-1)\Delta X}^{i\Delta X} F_z\left[1 + \frac{2}{3}\left(\frac{Z}{X}\right)^{\frac{3}{4}} + \frac{5}{9}\left(\frac{Z}{X}\right)^{\frac{3}{2}} + \frac{40}{81}\left(\frac{Z}{X}\right)^{\frac{9}{4}}\right.$$
$$\left. + \ldots\right]\mathrm{d}Z = \sum_{i=1}^{j}F_Z(j-1)\,\Delta X\left[\frac{1}{j-1}\right.$$
$$\left. + \frac{8}{21}\frac{i^{\frac{7}{4}}-(i-1)^{\frac{7}{4}}}{(j-1)^{\frac{7}{4}}} + \frac{2}{9}\frac{i^{\frac{5}{2}}-(i-1)^{\frac{5}{2}}}{(j-1)^{\frac{5}{2}}} + \ldots\right],$$

where

$$\bar{F}_z = \tfrac{1}{2}[F_{(i-1)\Delta X} + F_{i\Delta X}],$$
$$= \left[Y\bar{\theta}_i - \frac{\bar{\theta}_i^4}{N}\right].$$

Substitute this in equation (3) and rearrange some terms, to obtain

$$\theta_x = 1 + C_1 \frac{k_p}{k_f} X^{0\cdot5}\left[3\cdot530\left(\frac{1+\Phi}{N}\right) + \sum_{i=1}^{j}\left(Y\bar{\theta}_i - \frac{\bar{\theta}_i^4}{N}\right)S_{ij}^l\right], \quad (9)$$

where

$$S_{ij}^l = \frac{1}{j-1} + \frac{8}{21}\frac{i^{\frac{7}{4}} - (i-1)^{\frac{7}{4}}}{(j-1)^{\frac{7}{4}}} + \frac{2}{9} \times \frac{i^{\frac{5}{2}} - (i-1)^{\frac{5}{2}}}{(j-1)} + \frac{160}{1053}\frac{i^{\frac{13}{4}} - (i-1)^{\frac{13}{4}}}{(j-1)^{\frac{13}{4}}} + \dots$$

The equation (9) can be solved numerically for all values of $Z/X < 1$. However, for $Z/X = 1$ the equation in the form given is indeterminate. Once again use is made of beta function and in place of the summation, the integral is given by,

$$\int_0^X F_z\left[1 - \left(\frac{Z}{X}\right)^{\frac{3}{4}}\right]^{-\frac{2}{3}} dZ$$

$$Z = X \int_0^{X-\Delta X} F_z\left[1 - \left(\frac{Z}{X}\right)^{\frac{3}{4}}\right]^{-\frac{2}{3}} dZ + \bar{F}_z\left\{3\cdot530\,X - \int_0^{X-\Delta X}\left[1 - \left(\frac{Z}{X}\right)^{\frac{3}{4}}\right]^{-\frac{2}{3}} dZ\right\}.$$

In similar manner, the simplified equation for the plate temperature in turbulent flow is obtained, which is,

$$\theta_x = 1 + C_t \frac{k_p}{k_f} X^{0\cdot2}\left[9\cdot827\left(\frac{1+\Phi}{N}\right) + \sum_{i=1}^{j}\left(Y\bar{\theta}_i - \frac{\theta_i^4}{N}\right)S_{ij}^t\right], \quad (10)$$

where

$$S_{ij}^t = \frac{1}{j-1} + \frac{80}{171}\frac{i^{\frac{19}{10}} - (i-1)^{\frac{19}{10}}}{(j-1)^{\frac{19}{10}}} + \frac{160}{567}\frac{i^{\frac{14}{5}} - (i-1)^{\frac{14}{5}}}{(j-1)^{\frac{14}{5}}} + \dots$$

For $Z = X$, again the beta function is used to solve the integral.

3. DISCUSSION OF THE RESULTS

Comparison of the present results with those available in the literature

The general procedure to solve the above simplified governing equations is the method of iterations. To find a root of the equation $f(x) = 0$, the method of iterations is concerned with the finding of numbers $x_0, x_1, x_2, \dots, S$, which converge to limit S such that the equation $f(x) = 0$ is satisfied by $x = S$. Therefore an initial guess for the temperature profile of the plate is made. An accurate curve-fitting method needs to be used, because the temperature derivatives at the various locations of the plate length are obtained by differentiating the polynomial representing the assumed temperature profile. The initial guess for the temperature profile is then corrected to approach the temperature profile output obtained by the above numerical procedure. This process is continued till the input and output temperature profiles match within a prescribed accuracy to give a solution to the governing equation. The detailed numerical program is given in [10].

The results for the plate temperature in laminar and turbulent flows are presented in Figs. 2 and 3 respectively. For comparison, plots from the results of Sparrow and Lin [5] and Cess [6] are also shown. The ratio h_{RAD}/h_{UHF} in the analysis of Sparrow and Lin can be modified for comparison with the present results. The ratio h_{RAD}/h_{UHF} is a measure of the relative strengths of radiative and convective heat transfer. Therefore,

$$\frac{h_{\mathrm{RAD}}}{h_{\mathrm{UHF}}} = \frac{3{\cdot}511\,C_1}{N}\frac{k_p}{k_f}X^{0{\cdot}5}, \text{ for laminar flow,}$$

$$= \frac{10{\cdot}166\,C_t}{N}\frac{k_p}{k_f}X^{0{\cdot}2}, \text{ for turbulent flow.}$$

Also the parameter, $e/\epsilon\sigma T_\infty^4$ of Sparrow and Lin, is equivalent to $(1 + \Phi)$.

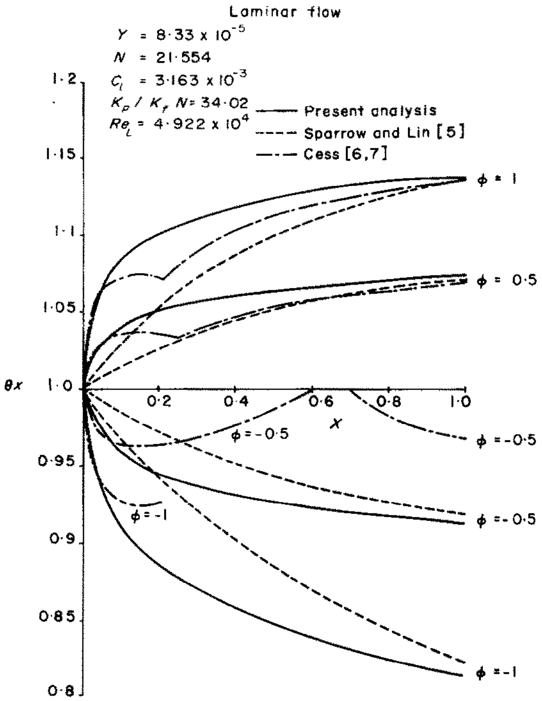

FIG. 2. Comparison of the plate temperature results in laminar flow.

From Figs. 2 and 3 it is seen that the results of Sparrow and Lin give a lower value of the plate temperature than the present results. The maximum deviation of about 5 per cent occurs for turbulent flow near $X = 0{\cdot}2$. The deviation subsequently decreases continuously. Both of the present methods (equations (7)–(10)) of solving the governing equation for plate temperature give identical results for all similar cases. Integration is essentially a summation process, which explains the identical nature of the results given by the present two methods.

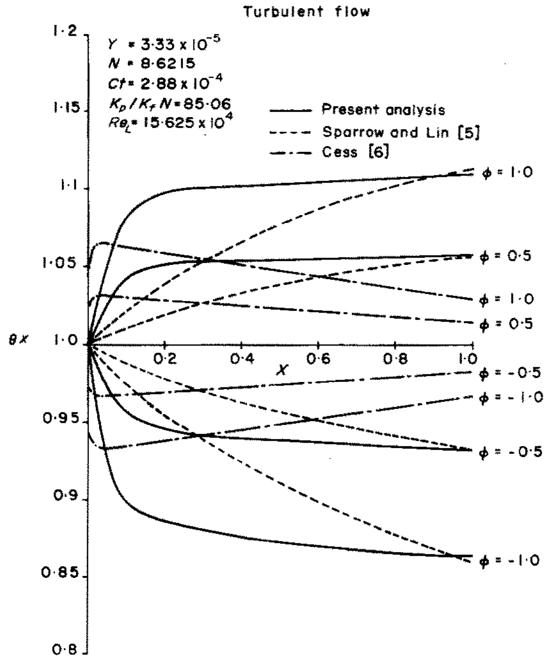

FIG. 3. Comparison of the plate temperature results in turbulent flow.

Therefore, the two methods have not been differentiated on the figures. The governing equations for the plate temperature in laminar and turbulent flows, equations (25) and (20) respectively given by Sparrow and Lin [5], are modified to correspond to the present analysis. For laminar flow, the analysis of Sparrow and Lin gives

$$\theta_x = 1 + \frac{h_{\mathrm{RAD}}}{h_{\mathrm{UHF}}}\left(\frac{e}{\epsilon\sigma T_\infty^4}\right) - \frac{1}{3X}\frac{h_{\mathrm{RAD}}}{h_{\mathrm{UHF}}} \times \int_0^x \frac{\theta_z^4}{(1 - Z/X)^{\frac{2}{3}}}\,\mathrm{d}Z,$$

$$= 1 + 3{\cdot}511\,C_1\frac{k_p}{k_f}X^{0{\cdot}5}\left\{\left(\frac{1+\Phi}{N}\right) + \sum_{i=1}^{j} \times\left(-\frac{\theta_i^4}{N}\right)\left[\left(1 - \frac{i-1}{j}\right)^{\frac{1}{3}} - \left(1 - \frac{i}{j}\right)^{\frac{1}{3}}\right]\right\}. \quad (11)$$

Including the conduction term also,

$$\theta_x = 1 + 3{\cdot}511\, C_1 \frac{k_p}{k_f} X^{0\cdot 5} \left\{ \left(\frac{1+\Phi}{N}\right) + \sum_{i=1}^{j} \times \left(Y\bar{\theta}_i - \frac{\theta_i^4}{N}\right)\left[\left(1 - \frac{i-1}{j}\right)^{\frac{1}{2}} - \left(1 - \frac{i}{j}\right)^{\frac{1}{2}}\right]\right\}. \quad (12)$$

Similarly, the two equations for turbulent flow without and with conduction heat transfer would be,

$$\theta_x = 1 + 10{\cdot}166\, C_t \frac{k_p}{k_f} X^{0\cdot 2} \left\{ \left(\frac{1+\Phi}{N}\right) + \sum_{i=1}^{j} \times \left(-\frac{\theta_i^4}{N}\right)\left[\left(1 - \frac{i-1}{j}\right)^{\frac{1}{9}} - \left(1 - \frac{i}{j}\right)^{\frac{1}{9}}\right]\right\} \quad (13)$$

and

$$\theta_x = 1 + 10{\cdot}166\, C_t \frac{k_p}{k_f} X^{0\cdot 2} \left\{ \left(\frac{1+\Phi}{N}\right) + \sum_{i=1}^{j} \times \left(Y\bar{\theta}_i - \frac{\theta_i^4}{N}\right)\left[\left(1 - \frac{i-1}{j}\right)^{\frac{1}{9}} - \left(1 - \frac{i}{j}\right)^{\frac{1}{9}}\right]\right\}. \quad (14)$$

The results of Sparrow and Lin in Figs. 2 and 3 are the solutions of equations (11) and (13) respectively by using a predictor–corrector numerical technique. But it is found that by using the method of iterations, the results of equations (11) and (13) are exactly the same as those for equations (7) or (9) and (8) or (10) respectively. This is not because of neglecting conduction terms in equations (11) and (13), since the conduction term is immaterial for the small value of thermal conductivity k_p used.

The results for $N = 0{\cdot}0$, $86{\cdot}2$ can hardly be distinguished in the graphs presented here. In the absence of conduction in the plate and because radiation heat exchange is negligible close to the leading edge, convection is the dominating mode of heat transfer near the leading edge. For convection, $T_x \propto 1/h_x$. Thus while h_x drops asymptotically from a very large to a finite value, the temperature would obviously show a sharp increase along the corresponding plate length for the case of small values of thermal conductivity. This effect is not obvious from the plots of Sparrow and Lin. As shown in the subsequent figures (i.e. Figs. 4, 7 and 8), the figures for lower thermal conductivity show a steeper profile as compared to the curves for higher thermal conductivity. Therefore even the results for the boundary condition $\dot{q}_0'' = qt$ (case of zero thermal conductivity) should show a sharp increase in temperature close to the leading edge. Nevertheless, it seems that the method of iterations with $\Delta X = 0{\cdot}01$ and the predictor–corrector method as employed by Sparrow and Lin, converge to different results.

Equation (5) of Cess [6] gives the plate temperature as

$$\theta_x = 1 + a_1\left(\Psi + \frac{a_2}{a_1}\Psi^2 + \ldots\right),$$

where a_1, $a_2, \ldots$ are the constants and for laminar flow Ψ is given by

$$\Psi = \frac{\epsilon\sigma T_\infty^3}{k_f}\sqrt{\left(\frac{\nu x}{U_\infty}\right)},$$

$$= \frac{k_p}{k_f}\frac{1}{N\sqrt{(Re_L)}}\sqrt{X}.$$

Therefore, for laminar flow,

$$\theta_x = 1 + \frac{\Phi}{0{\cdot}4059}\left[\frac{k_p}{k_f}\frac{1}{N\sqrt{(Re_L)}}(\sqrt{X}) - 8{\cdot}328\left(\frac{k_p}{k_f}\frac{1}{N\sqrt{(Re_L)}}\sqrt{X}\right)^2 + \ldots\right]. \quad (15)$$

and for turbulent flow,

$$\theta_x = 1 + \frac{\Phi}{0{\cdot}0273}\left[\frac{k_p}{k_f}\frac{1}{N(Re_L)^{0\cdot 8}}X^{0\cdot 2} - 142{\cdot}27\left(\frac{k_p}{k_f}\frac{1}{N(Re_L)^{0\cdot 8}}X^{0\cdot 2}\right)^2 + \ldots\right]. \quad (16)$$

For the plate temperature distribution in a laminar flow for higher values of Ψ, Cess [7] has given an alternate approach,

$$\theta_x = 1 + b_0\left[1 + \frac{b_1}{b_0}\Psi^{-1} + \ldots\right], \qquad (17)$$

where

$$b_0 = (1 + \Phi)^{\frac{1}{4}} - 1,$$

and

$$\frac{b_1}{b_0} = -\frac{0{\cdot}2927}{4(1 + \Phi)^{\frac{3}{4}}}.$$

With these substitutions, equation (17) becomes

$$\theta_x = (1 + \Phi)^{\frac{1}{4}} - \frac{0{\cdot}2927}{4}\frac{[(1 + \Phi)^{\frac{1}{4}} - 1]}{(1 + \Phi)^{\frac{3}{4}}} \times \frac{k_f}{k_p}\frac{N\sqrt{(Re_L)}}{\sqrt{X}} + \ldots. \qquad (18)$$

Results from equations (15), (16) and (18) are also shown in Figs. 2 and 3. The results of equation (15) converge only up to a small value of X around 0·1–0·2. The plots of equation (18) produce results which are physically consistent for $\Phi = 0{\cdot}5$ and 1·0, for $X = 0{\cdot}2$–1·0, but fail to give satisfactory results for $\Phi = -0{\cdot}5$. For turbulent flow also the equation (16) gives results which differ appreciably from the present results and those given by Sparrow and Lin. These plots show a steeper increase in the plate temperature near the leading edge for the same relations given by Cess [6] than the plots presented by Sparrow and Lin.

For both laminar and turbulent flows, the present two methods and the relations given by Sparrow and Lin, solved by using the present numerical technique give identical results, maximum deviation being less than 1 per cent. Therefore, to reduce the computer time, for all the curves presented henceforth, the method of iterations is used to solve the simpler equations (12) and (14), for laminar and turbulent flows respectively

Effect of various parameters

In most of the previous analyses, the conduction heat flow in the plate has been neglected.

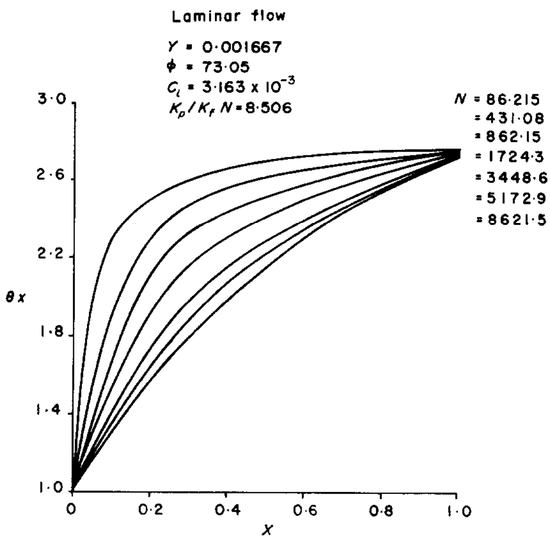

FIG. 4. Effect of thermal conductivity parameter N on the temperature of 0·254 mm thick plate in laminar flow.

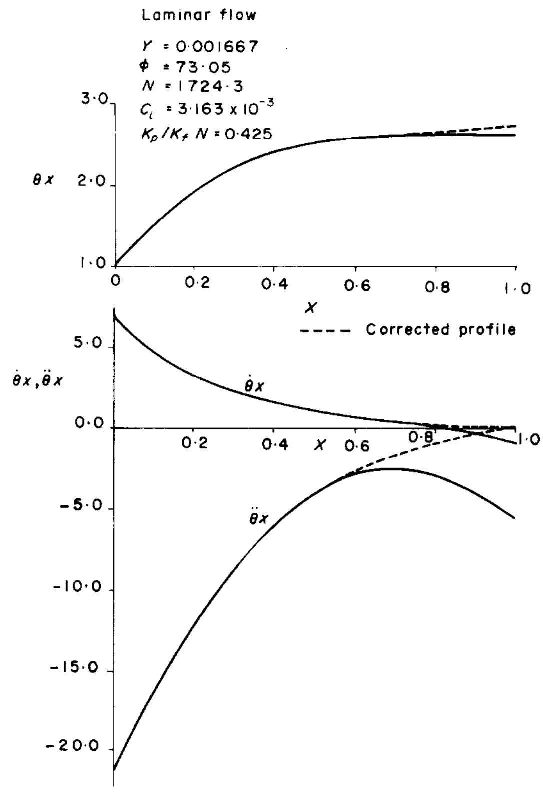

FIG. 5. Correction to be applied in the determination of $\ddot{\theta}_x$.

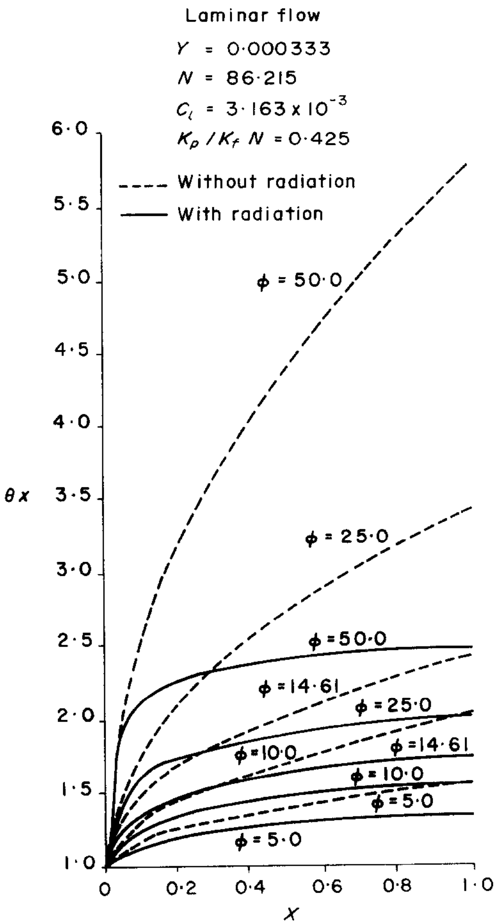

FIG. 6. Effect of the parameter Φ and radiation on the plate temperature in laminar flow.

Figure 4 shows the plate temperature for a wide range of thermal conductivity parameter N. It can be seen that the higher values of N bring down the plate temperature appreciably at smaller values of X, but with subsequent increase in X the effect of N continuously decreases. For a 0·254 mm ($Y = 0{\cdot}00166$) thick copper plate ($N \sim 1724{\cdot}3$) neglecting the thermal conductivity would cause an error of 62 per cent in the plate temperature at a location $X = 0{\cdot}05$ from the leading edge. For an infinitely large value of thermal conductivity, $\ddot{\theta}_x$ would approach zero, i.e. $\dot{\theta}_x$ would tend to a constant value.

Thus, in order to study the effect of thermal conductivity, numerical determination of $\ddot{\theta}_x$ becomes very critical. Because of the constant internal energy generation in the plate, its temperature has to increase with X. This means that $\dot{\theta}_x$ can never be negative, its value decreasing with x and becoming equal to zero in the extreme case. Therefore $\ddot{\theta}_x$ cannot have a point of inflection and its value has to approach close to zero at $X = 1$. The usual method of determining $\ddot{\theta}_x$ by differentiating the curve fit of the temperature profile is susceptible to some errors, which in turn may cause errors in the calculated temperature. This effect is shown in Fig. 5 by a solid curve in the range $X = 0{\cdot}7$–$1{\cdot}0$. Hence $\ddot{\theta}_x$ is forced to follow a physically more realistic profile as shown by the dashed line in Fig. 5.

Figure 6 shows the effect of changing the parameter Φ and also the plate temperature with and without inclusion of the radiative transfer term. Changing Φ essentially means changing either

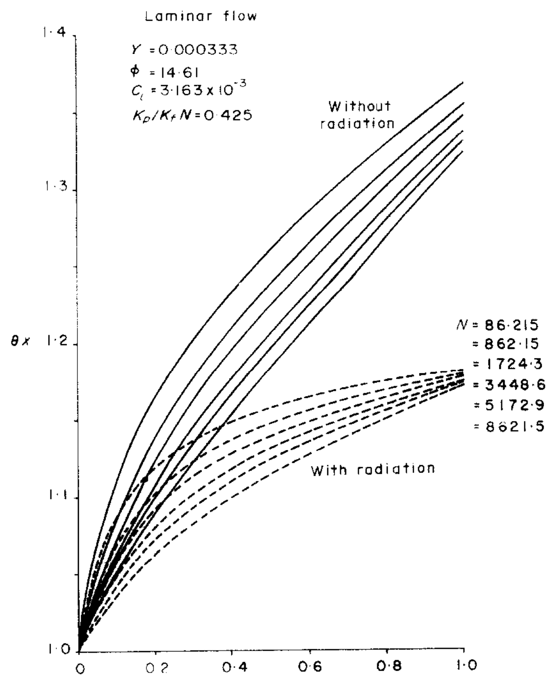

FIG. 7. Effection of the parameter N and radiation on the plate temperature in laminar flow.

internal energy generation or the plate thickness at low thermal conductivity. Thus, as would be expected, an increase in Φ increases the plate temperature as shown in Fig. 6. This figure also illustrates that radiation becomes more and more important with the increase in the plate temperature, which is caused by the higher volumetric energy input.

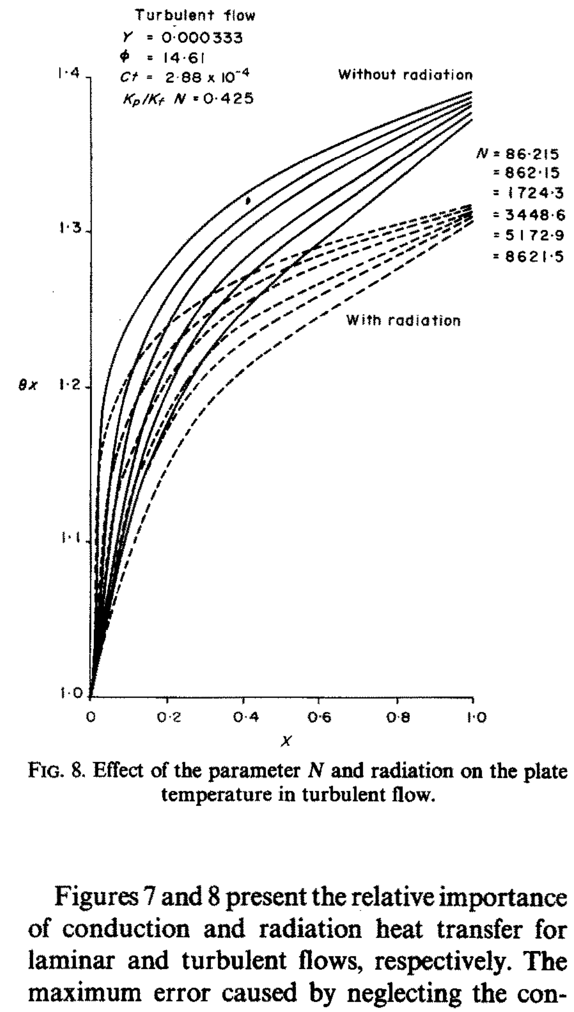

FIG. 8. Effect of the parameter N and radiation on the plate temperature in turbulent flow.

Figures 7 and 8 present the relative importance of conduction and radiation heat transfer for laminar and turbulent flows, respectively. The maximum error caused by neglecting the conduction heat transfer in a 0·0508 mm thick plate with $N = 1724{\cdot}3$ is about 10 per cent, as shown in Fig. 7. For the same volumetric energy generation, corresponding error for a 0·254 mm thick plate from Fig. 4, is 62 per cent. Absence of radiation could mean a maximum increase of about 43·5 per cent in the plate temperature in a laminar flow, the maximum increase occurring at the trailing edge of the plate when $N = 86{\cdot}215$. The corresponding increase in a turbulent flow flow is only about 5·55 per cent. This emphasizes the relative importance of radiative heat transfer in laminar and turbulent flows. Clearly, in turbulent flow convective heat transfer is much more dominant. Cess [6] has also reported that above a turbulent Reynolds number of $3{\cdot}5 \times 10^5$ the error caused by neglecting radiation effects would be less than 5 per cent.

4. CONCLUSIONS

The present work presents solutions for the plate temperature, where conduction heat transfer along the length of the plate becomes important. These are the cases of plates having a higher value of thermal conductivity and certain cases of thicker plates, where the assumption of constant temperature across the plate thickness can be maintained. This paper also gives solutions for integral equations which contain a delay factor of the more general type $[1 = (Z/X)^a]^{-b}$ in the kernel, though these solutions show little difference over the results using the delay factor of the type $[1 - (Z/X)]^{-b}$ as done by Sparrow and Lin.

The present results obtained by using the method of iterations, compare within 5 per cent of the results of Sparrow and Lin. Due to a sharp decrease in the value of convective heat transfer coefficient on moving away from the leading edge of the plate, there has to be a sharp increase in the plate temperature, unlike the plots of Sparrow and Lin. The relations given by Cess for plate temperature are not physically consistent over all ranges of parameters, as they do not give a constant increase (or an asymptotic approach to a constant value) in temperature in some cases. The neglect of radiation heat transfer in turbulent flow does not cause as severe an error as in laminar flow, a phenomenon also shown by Cess.

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DETERMINATION DE LA TEMPERATURE D'UNE PLAQUE DANS LE CAS D'ECHANGE DE CHALEUR COMBINE PAR CONDUCTION, CONVECTION ET RAYONNEMENT

Résumé—On a résolu numériquement par la méthode itérative l'équation intégro-différentielle singulière pour la température d'une plaque plane avec une génération d'énergie interne et un fluide qui s'écoule sur l'une de ses faces. Les résultats présentés se comparent bien à ceux de Sparrow et Lin, sauf pour la région du bord d'attaque de la plaque. On voit aussi que les relations données par Cess pour des problèmes similaires risquent de ne pas donner de solutions convergentes dans tous les cas. L'importance de la conduction dans une plaque à conductivité thermique élevée et du rayonnement dans le cas d'écoulement similaire a été démontrée.

BESTIMMUNG DER PLATTENTEMPERATUR IM FALL DES KOMBINIERTEN WÄRME-AUSTAUSCHS DURCH LEITUNG, KONVEKTION UND STRAHLUNG.

Zusammenfassung—Die Integro-Differentialgleichung für die Temperatur einer ebenen Platte mit inneren Wärmequellen, wobei ein Fluid über eine Plattenfläche strömt, wird numerisch durch Iteration gelöst. Die Ergebnisse lassen sich gut mit denen von Sparrow und Lin vergleichen, außer für den Plattenanfang. Man sieht auch, daß die Gleichung von Cess für ähnliche Probleme nicht für alle Fälle übereinstimmende Lösungen ergibt. Auch wurde die Wichtigkeit der Leitung in einer Platte von hoher thermischer Leitfähigkeit und die der Strahlung im Fall einer laminaren Strömung gezeigt.

ОПРЕДЕЛЕНИЕ ТЕМПЕРАТУРЫ ПЛАСТИНЫ В СЛУЧАЕ СЛОЖНОГО ТЕПЛООБМЕНА ТЕПЛОПРОВОДНОСТЬЮ, КОНВЕКЦИЕЙ И РАДИАЦИЕЙ

Аннотация—Методом итераций проведено численное решение сингулярного интегро-дифференциального уравнения для темпратуры пластины с внутренним источником энергии при обтекании ее с одной стороны. Результаты этой работы хорошо согласуются с даннвми Спэрроу и Лина за исключением передней кромки пластины. Установлено, что соотношения, приведенные Сессом для задач такого типа не всегда дают сходящиеся решения. Показано, что для пластины с высоким коэффициентом теплопроводности и при наличии излучения в случаях ламинарного обтекания теплопроводность играет существенную роль.